Input-output relations for multiport ring cavities

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Abstract. Quantum input-output relations for a generic n-port ring cavity are obtained by modeling the ring as a cascade of n interlinked beam splitters. Cavity response to a beam impinging on one port is studied as a function of the beam-splitter reflectivities and the internal phase-shifts. Interferometric sensitivity and stability are analyzed as a function of the number of ports.

Multiport ring cavities represent the natural generalization of two-port Fabry-Perot interferometers [1] to several modes of the radiation field. They find application in advanced interferometry, division multiplexing and optical cross connect. Recently, a three-port fiber ring laser was suggested and demonstrated to improve sensing resolution [2], whereas multiport optical circulators have been used for interconnecting single-fiber bidirectional ring networks [3]. In addition, a three-port reflection grating was demonstrated [4] and the corresponding input-output relations have been derived [5]. From a more fundamental perspective, multiport couplers, either multiport beam splitters or ring cavities, are crucial devices to generate and engineer multiphoton entangled states [6]. In fact, the cavity response is linear in the input modes for both kind of devices, with ring cavities offering the additional feature of a high nonlinearity with respect to the internal phase-shifts of the cavity. The use of ring cavities, supplemented by nonlinear media, has been also suggested to realize nondemolitive measurement and photon filtering [7].

In this letter, fully quantum input-output relations for a generic n-port ring cavity are obtained by modeling the ring as a cascade of n interlinked, suitably matched, beam splitters. In this way, the cavity response to an impinging beam, as well as the use of the cavity in interferometry, can be evaluated as a function of the beam-splitter reflectivities, the internal phase-shifts, and the number of ports.

Let us first illustrate the results in details for the case of a three-port ring cavity, which has been schematically depicted in Fig. 1. We assume that the three beam splitters used to build the cavity have the same transmissivity τ. We also assume that losses at the beam splitters are negligible. The reflectivity of each coupler is thus given by $\rho = 1 - \tau$. The input-output relation for the three beam splitters are given by
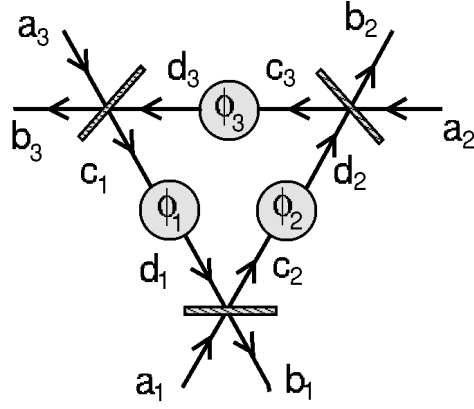
$$BS_k : \quad \begin{cases} b_k & = \tau^{\frac{1}{2}} d_k + \rho^{\frac{1}{2}} a_k \\ c_{1 \oplus k} & = -\rho^{\frac{1}{2}} d_k + \tau^{\frac{1}{2}} a_k \end{cases} \tag{1}$$

with $k = 1, 2, 3$ and $\oplus$ denoting sum modulo 3. Any additional phase-shift at the beam splitters may be absorbed into the internal phase-shifts ϕ_k. In order to build the cavity the matching relations $d_k = e^{i\phi_k} c_k$ should be also satisfied, together with

Eqs. (1). After lengthy but straightforward calculations one arrives at the input-output relations for the cavity

$$b_1 = \frac{1}{A_3}\left\{\sqrt{\rho}\left[1+\sqrt{\rho}e^{i\phi}\right]a_1 - \tau\sqrt{\rho}e^{i\phi_{13}}a_2 + \tau e^{i\phi_1}a_3\right\}$$

$$b_2 = \frac{1}{A_3}\left\{\tau e^{i\phi_2}a_1 + \sqrt{\rho}\left[1+\sqrt{\rho}e^{i\phi}\right]a_2 - \tau\sqrt{\rho}e^{i\phi_{12}}a_3\right\}$$

$$b_3 = \frac{1}{A_3}\left\{-\tau\sqrt{\rho}e^{i\phi_{23}}a_1 + \tau e^{i\phi_3}a_2 + \sqrt{\rho}\left[1+\sqrt{\rho}e^{i\phi}\right]a_3\right\} ,\qquad(2)$$

where $\phi_{jk} = \phi_j + \phi_k$, $\phi = \phi_1 + \phi_2 + \phi_3$ and $A_3 = 1 + \rho^{\frac{3}{2}}e^{i\phi}$. Unitarity of the mode transformations (2) can be explicitly checked through the normalization of the output modes $[b_j, b_k^\dagger] = \delta_{jk}$.

Figure 1. Three-port ring cavity as a cascade of three interlinked beam splitters. The cavity is built by three beam splitters with equal transmissivity τ. The matching relations $d_k = e^{i\phi_k}c_k$ should be satisfied together with the input-output relations (1) for each beam splitter.
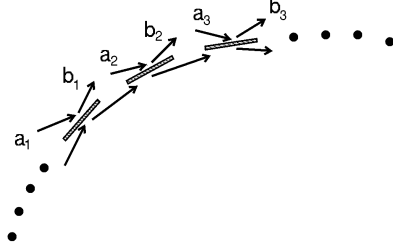
The above model can be generalized to a cavity with an arbitrary number of ports, see Fig. 2. We have

$$b_k^{(n)} = \sum_{j=1}^{n} M_{kj}^{(n)} a_j \qquad(3)$$

where

$$M_{11}^{(n)} = \frac{1}{A_n}\sqrt{\rho}\left[1 + (-)^{1+n}\rho^{\frac{n}{2}-1}e^{i\phi}\right]$$

$$M_{1j}^{(n)} = \frac{1}{A_n}\left\{(-)^n\,\tau\,\rho^{\frac{n}{2}}\sum_{j=2}^{n}(-)^j\rho^{-\frac{j}{2}}e^{i\theta_{1j}^{(n)}}\right\}\qquad(4)$$

and cyclic transformations for M_{kj}, $k = 2,...,n$, with $\phi = \sum_{k=1}^{n}\phi_k$, $A_n = 1 + \rho^{\frac{n}{2}}(-1)^{1+n}e^{i\phi}$ and $\theta_{tj}^{(n)} = \phi_t + \sum_{k=j+1}^{n}\phi_k$.

Figure 2. Multiport ring cavity as a cascade of interlinked beam splitters.

Explicitly, for a four-port cavity we have

$$b_1^{(4)} = \frac{1}{A_4} \left\{ \sqrt{\rho} \left[1 - \rho e^{i\phi}\right] a_1 + \tau \rho e^{i\phi_{134}} a_2 - \right.$$
$$\left. \tau \sqrt{\rho} e^{\phi_{14}} a_3 + \tau e^{i\phi_1} a_4 \right\} \tag{5}$$

and cyclic transformations, where $\phi_{134} \equiv \theta_{12}^{(4)} = \phi_1 + \phi_3 + \phi_4$.

Let us now consider the situation in which one of the port (say, port 1) is fed by a coherent beam $|\alpha\rangle$, whereas the other ports are left unexcited. Using mode transformations (3) one may analyze the cavity response to a given excitation, *i.e.* how the input mean energy $\langle a_1^\dagger a_1 \rangle \equiv \langle \alpha | a_1^\dagger a_1 | \alpha \rangle = |\alpha|^2$ is distributed among the n output photocurrents $I_k^{(n)} = b_k^{(n)\dagger} b_k^{(n)}$, $k = 1, ..., n$ obtained by detecting light at the n output ports of the cavity. Being the mode transformations linear also the output beams are coherent state with amplitudes $|\beta_k^{(n)}\rangle$. Upon defining the cavity response as

$$f_k^{(n)}(\rho, \phi) = \frac{\langle b_k^{\dagger(n)} b_k^{(n)} \rangle}{\langle a_1^{\dagger(n)} a_1^{(n)} \rangle},$$

one has $\beta_k^{(n)} = \alpha \sqrt{f_k^{(n)}} \exp\{i\theta_k^{(n)}\}$ with $\theta_k^{(n)} = \arg M_{k1}^{(n)}$ and
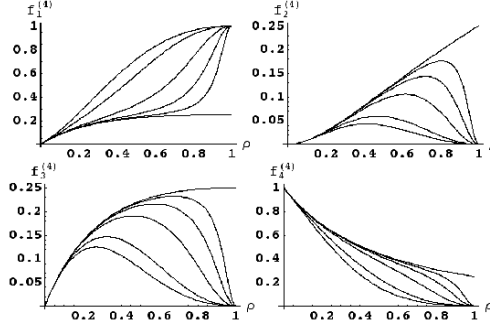
$$f_1^{(n)} = \frac{\rho}{|A_n|^2} \left[1 + \rho^{n-2} + 2(-)^{1+n} \cos\phi \, \rho^{\frac{n}{2}-1}\right] \tag{6}$$

$$f_k^{(n)} = \frac{(1-\rho)^2}{|A_n|^2} \rho^{n-k} \qquad 2 \le k \le n \tag{7}$$

where $|A_n|^2 = 1 + \rho^n + 2(-)^{1+n} \rho^{\frac{n}{2}} \cos\phi$. The cavity response explicitly depends on the mirror reflectivity ρ, while, remarkably, it depends on the internal phase-shifts ϕ_k only through the total phase-shift ϕ. The following sum-rule holds

$$f_1^{(n)} + \sum_{k=2}^{n} f_k^{(n)} = 1 \qquad \forall n, \forall \phi, \forall \rho,$$

which, in turn, assures energy conservation. In Fig. 3 we show the cavity responses $f_k^{(n)}$ for $n = 4$ as a function of the mirror reflectivity for different values of the total internal phase-shift. Notice that $0 \le f_k^{(4)} \le 1$ for $k = 1, 4$ and $0 \le f_k^{(4)} \le \frac{1}{4}$ otherwise.

Figure 3. Cavity responses $f_k^{(4)}$, $k = 1, .., 4$ of a four-port ring-cavity as a function of the mirrors' reflectivity for different values of the total internal phase-shift. In plot of $f_1^{(4)}$ ($f_k^{(4)}$ for $k \neq 1$), from bottom to top (from top to bottom) the curves corresponding to $\phi = 0, \frac{\pi}{20}, \frac{\pi}{10}, \frac{\pi}{5}, \frac{\pi}{2}, \pi$ respectively.

In general, the minimum of the cavity response at the first port is achieved for $\phi = 0$ for n even and for $\phi = \pi$ for n odd. For these values (cavity at resonance) we have

$$f_1^{(n)} = \rho \left(\frac{1 - \rho^{\frac{n}{2}-1}}{1 - \rho^{\frac{n}{2}}} \right)^2 \tag{8}$$

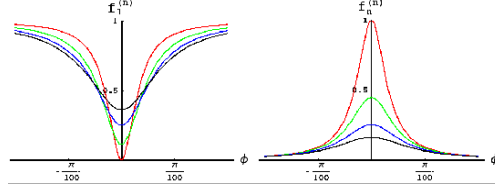$$f_k^{(n)} = \rho^{n-k} \left(\frac{1 - \rho}{1 - \rho^{\frac{n}{2}}} \right)^2 \quad 2 \leq k \leq n , \tag{9}$$

either for n even or odd. In the high-reflectivity limit $\rho \to 1$ we have $f_1^{(n)} = (1 - \frac{2}{n})^2$ and $f_k^{(n)} = \frac{4}{n^2}$ $\forall k \neq 1$. In other words, in a two-port cavity at resonance the energy is completely transferred to the second mode, while increasing the number of ports the energy is unavoidably "more distributed". For large n the input beam is mostly reflected on the beam $b_1^{(n)}$ and the cavity becomes opaque. An equal distribution at the output is obtained for $n = 4$ ($f_k^{(4)} = \frac{1}{4}$). The cavity response $f_1^{(n)}$ at the first port (last port $f_n^{(n)}$ respectively) monotonically increases (decreases) as the mirror reflectivity approaches unit value. On the other hand, $f_k^{(n)}$, $\forall k \neq 1, n$ show a maximum value, whose location depends on the internal phase-shift, as well as the number of ports of the cavity.

The sensitivity of the cavity in detecting perturbations to the internal phase-shift decreases as the number of ports increases. This is true either monitoring the cavity output at resonance or doing the same at a fixed working point in an interferometric setup. In Fig. 4 we show the cavity responses $f_1^{(n)}$ and $f_n^{(n)}$ as a function of the internal phase-shift for different number of ports. As it is apparent from the plot the curves flatten as the number of ports increases. The full-width half-minimum (maximum) of $f_1^{(n)}$ ($f_n^{(n)}$), for a generic value of n, is given by

$$\delta\phi_{\text{HW}}^{(n)} \simeq \frac{1 - \rho^{n/2}}{2\rho^{n/4}} \overset{\rho \to 1}{\simeq} \frac{n}{4}(1 - \rho) , \tag{10}$$

showing a linear increases of the half-width.

More generally, if one aims to detect the fluctuations of the internal phase-shift around a fixed working point $\phi = \phi^*$ by monitoring the output photocurrents $I_k^{(n)}$

Figure 4. Cavity responses $f_1^{(n)}$ (left) and $f_n^{(n)}$ (right) for a multiport ring cavity as a function of the internal phase-shift ϕ. The responses for $n = 2, 3, 4, 5$ and $\rho = 0.99$ are reported. The smaller is n, the peaked are the curves.
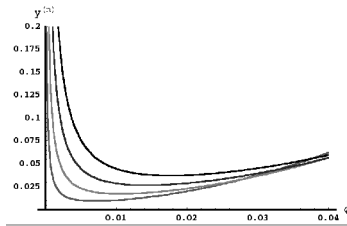
then the minimum detectable fluctuation corresponds to the quantity [8]

$$\delta\phi_k^{(n)} = \left| \left(\frac{\delta\langle I_k^{(n)} \rangle}{\delta\phi} \right)_{\phi=\phi^*} \right|^{-1} \sqrt{\langle \Delta I_k^{(n)2} \rangle} \,, \tag{11}$$

where $\langle \Delta I_k^{(n)2} \rangle = \langle (b_k^{(n)\dagger} b_k^{(n)})^2 \rangle - \langle b_k^{(n)\dagger} b_k^{(n)} \rangle^2$ denote the rms fluctuations of the output photocurrents. When a single input port is excited in a coherent state $\alpha\rangle$ also the output signals are coherent and Eq. (11) rewrites as

$$\delta\phi_k^{(n)} = \frac{\sqrt{f_k^{(n)}}}{|\alpha|} \left| \left(\frac{\partial f_k^{(n)}}{\partial\phi} \right)_{\phi=\phi^*} \right|^{-1} \,, \tag{12}$$

where $|\alpha|$ corresponds to the square root of the incoming average number of photons. The optimal working point ϕ^*, corresponding to maximum sensitivity, is the internal phase-shift that minimizes the value of $\delta\phi_k^{(n)}$ We found that ϕ^* is close, but not equal, to $\phi = 0$ for n even and to $\phi = \pi$ for n odd. Only slight differences are observed for different values of k, which vanishes for in the high-reflectivity regime. As a matter of fact, by increasing n the optimal working point ϕ^* moves away from $\phi = 0$ ($\phi = \pi$) and the minimum value of $\delta\phi$ increases. Since in the high-reflectivity regime $\rho \to 1$ the quantities $\delta\phi_k^{(n)}$ do not depends on k at fixed n, *i.e* $\delta\phi_k^{(n)} = \delta\phi_1^{(n)}$, $\forall k = 1, ..., n$ the overall sensitivity of the cavity may be evaluated as $\delta\phi^{(n)} = \delta\phi_1^{(n)}/\sqrt{n}$. In Fig. 5 we report the rescaled sensitivity $y^{(n)} = |\alpha|\delta\phi^{(n)}$ as function of ϕ for $\rho = 0.99$ and for different values of n. As it is apparent from the plot the overall sensitivity slightly degrades with increasing n, despite the factor $1/\sqrt{n}$ decreases. The curves versus ϕ flatten for increasing n and this implies that the need of tuning of the cavity at the optimal working point also becomes less stringent, *i.e* stability slightly increases.

Figure 5. Rescaled sensitivity $y^{(n)} = |\alpha|\delta\phi^{(n)}$ as function of working point ϕ for $\rho = 0.99$ and for different values of n. From bottom to top $n = 2, 3, 4, 5$.

In conclusion, by modeling a n-port ring-cavity as a cascade of n interlinked beam splitters we obtained its input-output relations in terms of the involved modes of the quantized radiation field. Using this approach, the cavity response to an impinging beam, as well as sensitivity to perturbations, can be straightforwardly evaluated as a function of the beam splitters reflectivity and the internal phase-shifts. We found that increasing the number of ports the input energy is unavoidably distributed over the output ports. The sensitivity of the cavity in detecting fluctuations of the internal phase-shift, either at resonance or at a fixed optimal working point, slightly degrades as the number of ports increases while, on the contrary, stability slightly increases.

References

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